Linear Operators on Matrices: The Invariance of Rank-k Matrices

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ABSTRACT

Let T be a linear operator on the space of all $m \times n$ complex matrices such that the rank of $T(A)$ is the rank of A whenever the rank of A is k. We show that there are nonsingular $m \times m$ and $n \times n$ matrices U and V respectively such that either

$$T(A) = UAV \qquad \text{for all } m \times n \text{ matrices } A,$$

or

$$m = n \quad \text{and} \quad T(A) = UA^tV \quad \text{for all } m \times m \text{ matrices } A.$$

1. INTRODUCTION

Let $\mathscr{M}_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ matrices over the algebraically closed field $\mathbb{F}$, and let $\rho(A)$ denote the rank of A. We define a *rank-k-preserver* to be a linear operator T on $\mathscr{M}_{m,n}(\mathbb{F})$ such that $\rho(A) = k$ implies $\rho(T(A)) = k$. Further, if T is a rank-k-preserver, we say that T *preserves rank-k matrices*. A linear operator which is a rank-k-preserver for each $k = 1, 2, \ldots, \min(m, n)$ is called a *rank preserver*.

Marcus and Moyls [4] proved in the following theorem in 1959:

THEOREM 1. *If $\mathbb{F}$ is algebraically closed and of characteristic 0 and T is a rank preserver, then there exist $m \times m$ and $n \times n$ matrices U and V,*

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respectively, such that either

$$\mathbb{T}(A) = UAV \quad \textit{for all} \quad A \in \mathscr{M}_{m,n}(\mathbb{F})$$

or

$$m = n \quad \textit{and} \quad \mathbb{T}(A) = UA^tV \quad \textit{for all } A \in \mathscr{M}_{m,m}(\mathbb{F}),$$

where A^t denotes the transpose of A.

Also in 1959, Marcus and Moyls [5] showed that if $\mathbb{T}$ is a rank-1 preserver then $\mathbb{T}$ is a rank preserver. Westwick [6] generalized these results to matrices over arbitrary algebraically closed fields in 1967. In a series of papers appearing between 1970 and 1983, Beasley [1–3] has shown that if $\mathbb{T}$ is a rank-k-preserver and $\max(m,n) \leqslant k+1$, $\min(m,n) = k$, $\max(m,n) \geqslant 3k/2$, or $\mathbb{T}$ is nonsingular, then $\mathbb{T}$ is a rank preserver (and hence has the form given above). This paper extends that result to all k without any restrictions except that the field must be the complex field. For easy reference we state:

THEOREM 2. *If $\mathbb{T}$ is a nonsingular rank-k-preserver on $\mathscr{M}_{m,n}(\mathbb{F})$, then T is a rank preserver.*

Proof. [1]. ∎

Let E be a set of nonnegative integers; then let $\mathscr{R}_E$ denote the subset of $\mathscr{M}_{m,n}(\mathbb{F})$ consisting of all matrices A with $\rho(A) \in E$. A subspace of $\mathscr{M}_{m,n}(\mathbb{F})$ which has only elements of rank k or 0 is called a *rank-k space*. This is equivalent to saying that the subspace is a subset of $\mathscr{R}_{\{k,0\}}$.

Throughout the remainder of this paper, we will adopt the convention that $k < m \leqslant n$. This causes no loss in generality. Further, we will assume that $\mathbb{F} = \mathbb{C}$, and use the notation $\mathscr{M}_{m,n}$ to denote $\mathscr{M}_{m,n}(\mathbb{C})$.

II. PRELIMINARY LEMMAS

LEMMA 1. *If there exists a singular rank-k-preserver on $\mathscr{M}_{m,n}$, then there exists a proper subset, G, of $\{1,2,\dots,k\}$, and a rank-k-preserver $\mathbb{T}$ such that $\operatorname{Im}\mathbb{T} \subseteq \mathscr{R}_G \cup \{0\}$ and $\operatorname{Ker}\mathbb{T} \subseteq \mathscr{R}_{G^c} \cup \{0\}$, where G^c denotes the complement of G in $\{1,2,\dots,m\}$.*

Proof. [3, Corollary 2] and [1, Theorem 9.1(iii)]. ∎

LEMMA 2. *The maximum dimension of a rank-k subspace of $\mathcal{M}_{m,n}$ is at most $m+n-2k+1$.*

Proof. [7]. ∎

LEMMA 3. *If* T *is an operator on $\mathcal{M}_{m,n}$ which preserves rank-k matrices, then* $\operatorname{Ker}\mathrm{T}\cap\mathcal{R}_{\{1\}}=\varnothing$.

Proof. Suppose that there is some rank-1 matrix in the kernel of T. Since T is a rank-k-preserver if and only $\mathrm{T}\circ\mathrm{S}$ is a rank-k-preserver for all rank preservers S, we may assume without loss of generality that $\mathrm{T}(E_{m,1})=0$, where E_{ij} is the matrix with a 1 in the (i,j) position and zero elsewhere. Define K_i by

$$K_i=\begin{bmatrix}0_{k,i-1} & I_k & 0_{k,n-k-i+1}\\ 0_{m-k,i-1} & 0_{m-k,k} & 0_{m-k,n-k-i+1}\end{bmatrix}$$

for $i=1,2,\dots,n-k+1$. Further, define L_i by

$$L_i=\begin{bmatrix}0_{k,i} & B_k & 0_{k,n-k-i}\\ 0_{m-k,i} & 0_{m-k,k} & 0_{m-k,n-k-i}\end{bmatrix}$$

for $i=1,2,\dots,n-k$, where $B_k=E_{11}+2E_{22}+3E_{33}+\cdots+kE_{kk}$, and let

$$C=\begin{bmatrix}0_{k-1,n-k+1} & I_{k-1}\\ 0_{m-k+1,n-k+1} & 0_{m-k+1,k-1}\end{bmatrix}.$$

Let $\mathcal{X}$ be the subspace of $\mathcal{M}_{m,n}$ generated by the above matrices, that is, $\mathcal{X}=\langle\{K_i\}_{i=1}^{n-k+1}\cup\{L_i\}_{i=1}^{n-k}\cup\{C\}\rangle$. First observe that the dimension of $\mathcal{X}$ is $2n-2k+2>m+n-2k+1$. Secondly, if $X\in\mathcal{X}$ then $X=\sum_{i=1}^{n-k+1}\alpha_iK_i+\sum_{i=1}^{n-k}\beta_iL_i+\gamma C$ for some choice of the α_i's, β_i's, and γ in $\mathbb{C}$. If X is some nonzero element of $\mathcal{X}$, then either X is a multiple of C or some α_i or β_i is nonzero. If X is a multiple of C, then $\rho(X)=k-1$ and $\rho(E_{m,1}+X)=k$, so that $\rho(\mathrm{T}(X))=\rho(\mathrm{T}(E_{m,1}+X))=\rho(E_{m,1}+X)=k$, since $\mathrm{T}(E_{m,1})=0$. If some α_i or β_i is nonzero, then $\rho(X)=k$ or $\rho(X)=k-1$. If $\rho(X)=k$

then $\rho(\mathrm{T}(X)) = k$, while if $\rho(X) = k-1$ then $\alpha_1 = 0$ and hence $\rho(E_{m,1}+X) = k$, so that $\rho(\mathrm{T}(X)) = \rho(\mathrm{T}(E_{m,1}+X)) = \rho(E_{m,1}+X) = k$. In every case, $\rho(\mathrm{T}(X)) = k$ whenever $X \in \mathscr{X}$ and $X \neq 0$. Thus, $\mathrm{T}(\mathscr{X})$ is a rank-k space of dimension $2n-2k+2$, a contradiction to Lemma 2. ∎

LEMMA 4. *If* T *is a rank-k-preserver on* $\mathscr{M}_{m,n}$, *then* $\operatorname{Ker}\mathrm{T} \cap \mathscr{R}_E = \varnothing$, *where* $E = \{m-k+1,\dots,k\}$.

Proof. If $n \geqslant 3k/2$, then $\operatorname{Ker}\mathrm{T} = \{0\}$ by [3], and the lemma follows. Thus we assume that $k < m \leqslant n < 3k/2$.

Suppose that some matrix of rank $s > m-k$ is in the kernel of T. We may assume without loss of generality that

$$A = \begin{bmatrix} 0 & I_s \\ 0 & 0 \end{bmatrix} \in \operatorname{Ker}\mathrm{T}.$$

Define P_i, $i = 1,2,\dots,n-k+1$, by

$$P_i = \begin{bmatrix} 0_{m-k,i-1} & 0_{m-k,k} & 0_{m-k,n-k-i+1} \\ 0_{k,i-1} & I_k & 0_{k,n-k-i+1} \end{bmatrix},$$

and define Q_i, $i = 1,2,\dots,m-k+1$, by

$$Q_i = \begin{bmatrix} 0_{s+i-1,k-s} & 0_{s+i-1,n-k+s} \\ I_{k-s} & 0_{k-s,n-k+s} \\ 0_{m-k-i+1,k-s} & 0_{m-k-i+1,n-k+s} \end{bmatrix}.$$

Now, let $\mathscr{X} = \langle \{P_i\}_{i=1}^{n-k+1} \cup \{Q_i\}_{i=1}^{m-k+1} \rangle$, the subspace of $\mathscr{M}_{m,n}$ generated by the above matrices. Clearly, the dimension of $\mathscr{X}$ is $m+n-2k+2$.

Now if $X \in \mathscr{X}$, then either $\rho(X) = k$, whenever the expression of X as a linear combination of the P_i's and Q_i's contains a nonzero multiple of some P_i; $\rho(X) = k-s$, whenever X contains a nonzero multiple of some Q_i and no nonzero multiple of any P_i; or X is zero. If $\rho(X) = k$ then $\rho(\mathrm{T}(X)) = k$. If $\rho(X) = k-s$ then $\rho(A+X) = k$, and since $\mathrm{T}(A) = 0$, $\rho(\mathrm{T}(X)) = \rho(\mathrm{T}(A+X)) = k$ Thus if K is a nonzero element of $\mathscr{X}$, $\mathrm{T}(K)$ is nonzero and of rank k. That is, $\mathrm{T}(\mathscr{X})$ is a rank-k space of dimension $m+n-2k+2$, a contradiction to Lemma 2. ∎

Define $\mathscr{G}$ to be the subspace of $\mathscr{M}_{m,n}$ of all matrices of the form

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

where A is $k \times k$.

LEMMA 5. *If* $\mathbf{T}$ *is a singular rank-k-preserver, then there exists a rank preserver* $\mathbf{S}$ *such that* $\operatorname{Im} \mathbf{S} \circ \mathbf{T} \subseteq \mathscr{G}$.

Proof. [1, Lemma 2.2]. ■

III. THE MAIN THEOREM

We are now ready to prove the main theorem. For a matrix X and subsequences α and β of $\{1,2,\ldots,m\}$ and $\{1,2,\ldots,n\}$, respectively, we use the notation $X[\alpha|\beta]$ to denote the submatrix of X on rows α and columns β.

THEOREM 3. *If* $\mathbf{T}$ *is a rank-k-preserver on* $\mathscr{M}_{m,n}(\mathbb{C}) = \mathscr{M}_{m,n}$, *then there exist* $m \times m$ *and* $n \times n$ *matrices* U *and* V *respectively such that either*

$$\mathbf{T}(A) = UAV \qquad \textit{for all} \quad A \in \mathscr{M}_{m,n},$$

or

$$m = n \quad \textit{and} \quad \mathbf{T}(A) = UA^tV \quad \textit{for all } A \in \mathscr{M}_{m,m},$$

where A^t *denotes the transpose of* A.

Proof. Suppose that $\mathbf{T}$ is a singular rank-k-preserver. By [3] we may assume that $n < 3k/2$. By Lemma 1, we may assume that for some subset G of $\{1,\ldots,k\}$, $\operatorname{Ker}\mathbf{T} \subseteq \mathscr{R}_{G^c} \cup \{0\}$ and $\operatorname{Im}\mathbf{T} \subseteq \mathscr{R}_G \cup \{0\}$. Note that $\mathbf{T}$ is 1-1 and onto $\operatorname{Im}\mathbf{T}$. Let s be the largest integer in G^c which is less than k. By Lemma 4, $s \leqslant m - k$. Without loss of generality we may assume that

$$A = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} \in \operatorname{Ker}\mathbf{T}.$$

Let $\mathscr{H}$ be the rank-1 space $\mathscr{H} = \langle \{E_{k+1,k+i}\}_{i=1}^{n-k} \rangle$. By Lemma 3, $\mathrm{T}(H) \neq 0$ for all nonzero $H \in \mathscr{H}$. Also by Lemma 5 we may assume that $\operatorname{Im}\mathrm{T} \subseteq \mathscr{G}$. Thus, $\mathscr{H} \cap \operatorname{Im}\mathrm{T} = \{0\}$.

Let $H \in \mathscr{H}$. Since $\mathrm{T}(H) \neq 0$, there is some $L \in \operatorname{Im}\mathrm{T}$ such that $\mathrm{T}(L) = \mathrm{T}(H)$. That is, $\mathrm{T}(L - H) = 0$. Since the coefficient of t^s in the expansion of $\det((tA + L - H)[1,2,\dots,s,k+1|1,2,\dots,s,k+i])$ is nonzero for some i corresponding to a nonzero entry in H, and since $tA + L - H$ is in the kernel of T, the rank of $tA + L - H$ is at least $s+1$ for some t; thus by the choice of s, the rank is at least $k+1$ for some t. Since the rank of A is s, the rank of $L - H$ is at least $k - s + 1 \geqslant k - (m-k) + 1 = 2k - m + 1 > s$. Thus $\rho(L - H) \geqslant k + 1$. Since $\rho(H) = 1$, and $\rho(L) \leqslant k$ (since $L \in \operatorname{Im}\mathrm{T}$), it follows that $\rho(L) = k$, and thus $\rho(\mathrm{T}(H)) = \rho(\mathrm{T}(L)) = k$.

For $i = 0,1,2,\dots,n-k$, define X_i to be

$$X_i = \begin{bmatrix} 0_{1,i} & 0_{1,k} & 0_{1,n-k-i} \\ 0_{k,i} & D_k & 0_{k,n-k-i} \\ 0_{m-k-1,i} & 0_{m-k-1,k} & 0_{m-k-1,n-k-i} \end{bmatrix}$$

where

$$D_k = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

is a $k \times k$ matrix. Further, let

$$Y = \begin{bmatrix} 0_{s+1,n-k+1} & 0_{s+1,k-s} & 0_{s+1,s-1} \\ 0_{k-s,n-k+1} & D_{k-s} & 0_{k-s,s-1} \\ 0_{m-k-1,n-k+1} & 0_{m-k-1,k-s} & 0_{m-k-1,s-1} \end{bmatrix},$$

and let $\mathscr{K} = \langle \{X_i\}_{i=1}^{n-k+1} \cup \{Y\} \cup \mathscr{H} \rangle$.

Now if $C \in \mathscr{K}$ and C is nonzero, then either $\rho(C) = k$, $\rho(C) = k - s$, or $C \in \mathscr{H}$. If $\rho(C) = k$ then $\rho(\mathrm{T}(C)) = k$. If $\rho(C) = k - s$ then $\rho(A + C) = k$, and since $\mathrm{T}(A + C) = \mathrm{T}(C)$, $\rho(\mathrm{T}(C)) = k$. If $C \in \mathscr{H}$, then from the above, $\rho(\mathrm{T}(C)) = k$. In any case, $\rho(\mathrm{T}(C)) = k$. Thus, $\mathrm{T}(\mathscr{K})$ is a rank-k space of dimension $2n - 2k + 2$, a contradiction to Lemma 2.

Thus T must be nonsingular. Applying Theorem 1 to Theorem 2 completes the proof of the theorem. ∎

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